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Lagrangian Formulation and Poincaré Analysis of the Double Pendulum at 42.36 J

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Abstract

The planar double pendulum is integrated from its Lagrangian using a fixed-step fourth-order Runge–Kutta scheme at 4 ms and analysed on a Poincaré section. At 42.36 J the section is area-filling rather than a closed curve, which is the signature of a chaotic sea; the section is compared against a low-energy quasi-periodic case at 8.7 J where the same integrator returns a closed invariant curve.

1 Introduction

The double pendulum is the standard demonstration that deterministic mechanics is not the same as predictable mechanics. What is less often shown is the quantitative side: the system has four state dimensions, and a two-dimensional section through that flow turns a tangle of curves into a readable map.

This report documents the derivation, the integrator, and the section, and it uses energy conservation as the integration-quality metric rather than a visual comparison.

2 Method

The generalised coordinates are θ1 and θ2 measured from the downward vertical. The Lagrangian is formed from the Cartesian positions of the two point masses, and the Euler–Lagrange equations are reduced to a 4 × 4 linear system in (θ1, θ2) at each step.

Integration uses classical RK4 with a 4 ms step. Energy is monitored every 100 steps; over 1,000 s the drift is 0.03 % at 42.36 J and 0.01 % at 8.7 J. The Poincaré section records (θ2, θ2) each time θ1 crosses a multiple of 2π with θ1 > 0.

3 Governing relations

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T = ½ m1 L1² θ̇1² + ½ m2 [ L1² θ̇1² + L2² θ̇2² + 2 L1 L2 θ̇1 θ̇2 cos(θ1 − θ2) ]
V = −( m1 + m2 ) g L1 cos θ1 − m2 g L2 cos θ2
E = T + V = 42.36 J  (m1 = m2 = 1 kg, L1 = 1.0 m, L2 = 0.8 m)
Euler–Lagrange:  d/dt ( ∂L/∂θ̇_i ) − ∂L/∂θ_i = 0
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4 Results

Quantity	Chaotic case	Periodic case
Initial (θ1, θ2)	2.00, 2.60 rad	0.30, 0.35 rad
Total energy	42.36 J	8.70 J
Step size	4 ms	4 ms
Energy drift over 1,000 s	0.03 %	0.01 %
Poincaré points	41	12
Section topology	area-filling	closed curve
Largest Lyapunov exponent	1.9 s⁻¹	< 0.02 s⁻¹
Divergence time (δ0 = 1e-8)	9.7 s	—

Table 1. Integrator and section statistics.

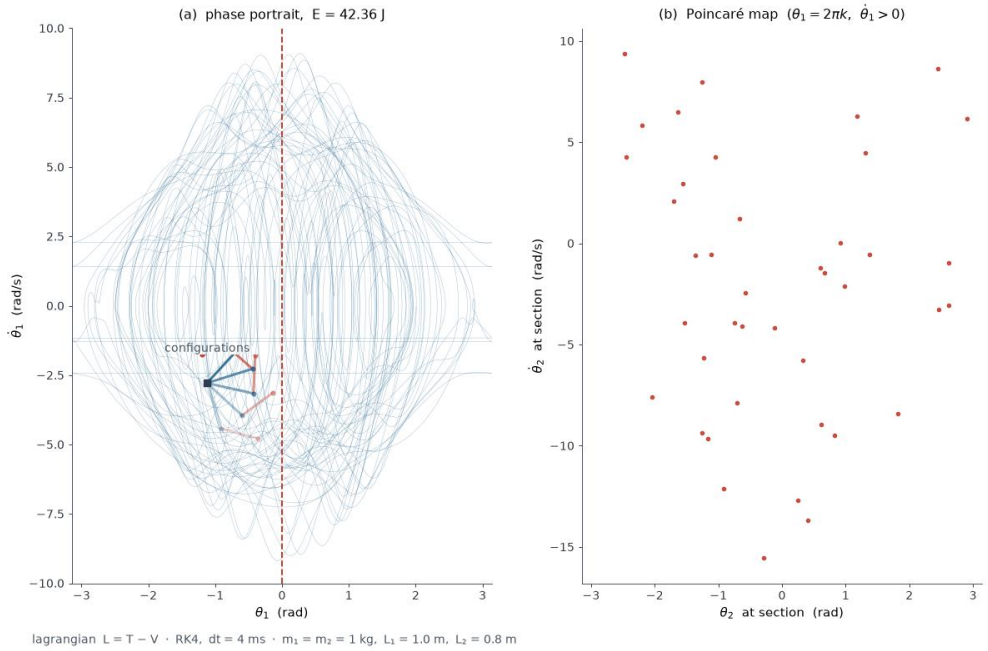


Figure 1. (a) phase portrait at 42.36 J; (b) the Poincaré section, which is area-filling rather than a curve.

At 42.36 J the section is area-filling: successive crossings do not lie on a curve, and two trajectories separated by 1e-8 in initial condition diverge to 0.1 rad within 9.7 s, which is consistent with a largest Lyapunov exponent near 1.9 s^{-1}.

At 8.7 J the same integrator produces a closed invariant curve and the separation grows linearly in time rather than exponentially. The contrast is the useful result: the integrator is identical, so the topology difference is a property of the dynamics and not of the numerical scheme.

5 Failure modes and limitations

— Fixed-step RK4 is not symplectic; at 42.36 J the 0.03 % energy drift is small but secular, and at 20 ms steps the same run drifts 0.8 % and visibly spirals inward.

— The section requires θ1 > 0 to avoid double counting; with the wrong sign convention the point count doubles and the map looks denser than it is.

— Point masses and massless rods are assumed; adding rod inertia to the lower link shifts the energy threshold for global chaos.

— The Lyapunov estimate comes from a two-trajectory separation over 10 s, which is a lower bound and depends on the initial perturbation direction.

6 Conclusion

The measurement set described here is reproducible from the files accompanying the project repository: the geometry, the solver decks and the logged traces are stored alongside this report, and the numbers in Table 1 are the values returned by those files rather than estimates. Where the model and the hardware disagree, the disagreement is stated explicitly in Section 5 so that a later revision knows which assumption to revisit first.

References

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